TRANSFORMATIONS OF CONJUGATE SYSTEMS WITH EQUAL INVARIANTS

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Recently we considered at length two types of transformations of conjugate systems of curves on surfaces, which we called[1,2] *transformations* K and *transformations* Ω. It is the purpose of this note to show that there is a fundamental relation connecting these transformations.

If x, y, z are the cartesian coördinates of a surface S, referred to a conjugate system with equal point invariants, it is necessary and sufficient that x, y, z are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0, \tag{1}$$

where ρ is a function of u and v in general. If θ_1 is any solution of (1) and λ_1 is the function defined by the consistent equations

$$\frac{\partial \lambda_1}{\partial u} = -\rho\frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \lambda_1}{\partial v} = \rho \frac{\partial \theta_1}{\partial v}, \tag{2}$$

the equations

$$\frac{\partial}{\partial u}(\lambda_1 x_1) = -\rho\left(x\frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial x}{\partial u}\right), \quad \frac{\partial}{\partial v}(\lambda_1 x_1) = \rho\left(x\frac{\partial \theta_1}{\partial v} - \theta_1\frac{\partial x}{\partial v}\right) \tag{3}$$

are consistent, and x_1 satisfies an equation of the form (1) when ρ is replaced by ρ_1, given by

$$\rho_1 = \lambda_1^2 / \rho\theta_1^2. \tag{4}$$

The three functions x_1, y_1, z_1, given by (3) and similar equations in the y's and z's, are the cartesian coördinates of a surface S_1, which by definition is in the relation of a transformation K with the surface S. If M and M_1 are corresponding points on these surfaces, the developables of the congruence G of the lines MM_1 cut S and S_1 in the parametric curves and the focal points on the line MM_1 are harmonic to M and M_1. Conversely, when two surfaces S and S_1 are so related that the congruence of lines joining corresponding points meet S and S_1 in conjugate systems and the points of S and S_1 are harmonic to the focal points of the congruence, they are in the relation of a transformation K, as defined analytically by (3) [cf. M_1, p. 403].

Whenever any two surfaces are in a one-to-one correspondence such that the developables of the congruence of lines joining corresponding points cut the surfaces in conjugate systems, the lines of intersection of corresponding tangent planes to these surfaces form a congruence G' whose developables correspond to the developables of the former congruence.[3] Evidently two surfaces in the relation of a transformation K possess this property, but it is not a characteristic property.

When any surface S is subjected to a polar transformation with respect to a quadric, points and tangent planes are transformed into tangent planes and points respectively of the new surface Σ. Since straight lines go into straight lines, it is readily shown that conjugate directions on S go into conjugate directions on Σ. Also a congruence of lines is transformed into a congruence of lines, and the developable surfaces of the two congruences correspond; furthermore to the focal points on a line correspond the focal planes through the corresponding line of the other congruence.

We consider the effect of applying a polar transformation to two surfaces S and S_1 in the relation of a transformation K. If the new surfaces be denoted by Σ and Σ_1, the lines joining corresponding points $\overline{M}$ and $\overline{M}_1$ on these surfaces form a congruence $\overline{G}'$ whose developables meet Σ and Σ_1 in conjugate systems, and the tangent planes to Σ and Σ_1 meet in the lines of a congruence $\overline{G}$ whose developables correspond to the developables of the congruence $\overline{G}'$; moreover, the focal planes of the congruence $\overline{G}$ are harmonic to the tangent planes to Σ and Σ_1. These properties of the surfaces Σ and Σ_1 are possessed likewise by a pair of surfaces in the relation of a transformation[2] Ω. It is our purpose to show that Σ and Σ_1 are in the relation of a transformation Ω, and that the properties just mentioned are characteristic of transformations Ω.

The equation of any quadric may be put in the form

$$ax^2 + by^2 + cz^2 + 2\,dx + 2\,ey + 2\,fz + g = 0. \qquad (5)$$

The equation of the polar plane of the point (x, y, z) with respect to this quadric is

$$Xx' + Yy' + Zz' = W. \qquad (6)$$

where x', y', z' are the current rectangular coördinates and

$$X = \frac{ax+d}{\sqrt{\sigma}}, \quad Y = \frac{by+e}{\sqrt{\sigma}}, \quad Z = \frac{cz+f}{\sqrt{\sigma}}, \quad W = -\frac{dx+ey+fz+g}{\sqrt{\sigma}},$$

$$\sigma = (ax+d)^2 + (by+e)^2 + (cz+f)^2. \qquad (7)$$

Hence X, Y, Z are the direction-cosines of the plane (6), and if x, y, z are the cartesian coördinates of a surface S, then X, Y, Z and W are the tangential coördinates of its transform Σ. These four coördinates are solutions of the equation

$$\frac{\partial^2\varphi}{\partial u\partial v}+\frac{\partial\log\sqrt{\overline{\rho}}}{\partial v}\frac{\partial\varphi}{\partial u}+\frac{\partial\log\sqrt{\overline{\rho}}}{\partial u}\frac{\partial\varphi}{\partial v}+F\varphi=0, \tag{8}$$

where

$$\sqrt{\overline{\rho}}=\sqrt{\rho\sigma},\quad F=\frac{1}{\sqrt{\sigma}}\frac{\partial^2\sqrt{\sigma}}{\partial u\partial v}+\frac{\partial\log\sqrt{\rho}}{\partial v}\frac{\partial\log\sqrt{\sigma}}{\partial u}+\frac{\partial\log\sqrt{\rho}}{\partial u}\frac{\partial\log\sqrt{\sigma}}{\partial v}. \tag{9}$$

Consequently the parametric conjugate system on Σ has equal tangential invariants.

When the same polar transformation is applied to S_1, the tangential coördinates of Σ_1, namely X_1, Y_1, Z_1, W_1, are obtained from (7) on replacing x, y, z by x_1, y_1, z_1 respectively. These functions satisfy an equation of the form (8), obtained on replacing ρ and F by ρ_1 and F_1, where in consequence of (4),

$$\sqrt{\overline{\rho}_1}=-\lambda_1\sqrt{\sigma_1}\Big/\sqrt{\rho}\,\theta_1,\quad \sigma_1=(ax_1+d)^2+(by_1+e)^2+(cz_1+f)^2, \tag{10}$$

and F_1 is analogous to F.

Because of equations (3), the functions X_1, Y_1, Z_1, W_1 are the respective transforms of X, Y, Z, W by means of the equations

$$\frac{\partial}{\partial u}(\lambda_1\sqrt{\sigma_1}\varphi_1)=-\rho\sigma\left(\varphi\frac{\partial w_1}{\partial u}-w_1\frac{\partial\varphi}{\partial u}\right),\ \frac{\partial}{\partial v}(\lambda_1\sqrt{\sigma_1}\varphi_1)=\rho\sigma\left(\varphi\frac{\partial w_1}{\partial v}-w_1\frac{\partial\varphi}{\partial v}\right), \tag{11}$$

where

$$w_1=\theta_1/\sqrt{\sigma}. \tag{12}$$

Consequently w_1 is a solution of (8). From (9) and (10) it follows that

$$\lambda_1\sqrt{\sigma_1}=-\sqrt{\overline{\rho}\,\overline{\rho}_1}\,w_1. \tag{13}$$

Hence equations (11) are equivalent to those of a transformation[2] Ω.

We are now in a position to prove the theorem:

If Σ and Σ_1 are so related that for the congruence of lines of intersection of corresponding tangent planes π and π_1 to Σ and Σ_1 respectively the focal planes of the congruence are harmonic to π and π_1, and the developables of the congruence correspond to conjugate systems on Σ and Σ_1, the latter systems have equal tangential invariants; and Σ and Σ_1 are in the relation of a transformation Ω. For if we apply the polar transformation to the surfaces Σ and Σ_1, the resulting surfaces are related in the manner which

we have stated to be characteristic of a transformation K. But as we have just shown, the surfaces Σ and Σ_1, being polar transforms of two surfaces in the relation of a transformation K, are themselves in the relation of a transformation Ω. Hence we have proved the above theorem and also the following:

When two surfaces S and S_1 are in the relation of a transformation K, their polar transforms are in the relation of a transformation Ω; and conversely.

Because of the dual relation between these two types of transformations, we are enabled to add to Theorems 4 and 6 of memoir M_1 the dual of the last part of the theorem of §4 of memoir M_2, and thus have the following theorem of permutability of the transformations K: *If S_1 and S_2 are two surfaces arising from S by transformations K, there can be found by quadratures an infinity of surfaces S', each of which is in the relation of transformations K with both S_1 and S_2. If M, M_1 and M_2 denote corresponding points on S, S_1 and S_2, the corresponding points M' on the surfaces S' lie on a line through M and in the plane π of the points M, M_1, M_2. The corresponding tangent planes to the surfaces S' envelope a quadric cone to which are tangent the tangent planes to S, S_1, S_2 at M, M_1, M_2. Moreover, the plane π touches its envelope at the point of intersection P of the lines MM' and M_1M_2; and the parametric curves on the envelope form a conjugate system whose tangents are harmonic to the lines MM' and M_1M_2, and contain the focal points of the lines MM_1, MM_2, $M'M_1$, $M'M_2$.* An analogous theorem of permutability of transformations Ω follows from the above in accordance with the principle of duality.

The relation between the two types of transformations is likewise helpful in interpreting the significance of certain evident forms of the transforming functions θ_1 and w_1. Thus we have shown (M_1, p. 401) that the necessary and sufficient condition that for two surfaces S and S_1 in the relation of a transformation K the corresponding tangent planes be parallel is that θ_1 be constant. In this case S and S_1 are associate surfaces, that is not only are the tangent planes parallel but also to asymptotic lines on either surface corresponds a conjugate system on the other. Moreover, any two associate surfaces are in this special kind of relation of a transformation K. Let S and S_1 be two associate surfaces and apply to them the polar transformation with respect to the quadric (5), where a, b, c are different from zero. Since corresponding tangent planes to S and S_1 meet in a line in the plane at infinity, the lines joining corresponding points on Σ and Σ_1 meet in a point—the pole of the plane at infinity with respect to the quadric. From (7)

it is seen that the coördinates of this point are $-d/a$, $-e/b$, $-f/c$. Conversely, if the lines joining corresponding points on two surfaces Σ and Σ_1, in the relation of a transformation Ω, meet in a point M, the surfaces S and S_1 arising from Σ and Σ_1 by a polar transformation are so placed that the lines of intersection of corresponding tangent planes to S and S_1 lie in a plane, the polar plane of M. When in particular, the fundamental quadric of the transformation is chosen so that M is the pole of the plane of infinity (which can always be done) S and S_1 are associate surfaces.

When θ_1 is a constant, it follows from (12) that w_1 is equal to $1/\sqrt{\sigma}$ to within a constant multiplier, which is unessential, as is evident from (11) and (13). From (7) we have

$$W + \frac{d}{a}X + \frac{e}{b}Y + \frac{f}{c}Z = \frac{1}{\sqrt{\sigma}}\left(\frac{d^2}{a} + \frac{e^2}{b} + \frac{f^2}{c} - g\right). \tag{14}$$

We choose the quadric so that the coefficient in (14) of $1/\sqrt{\sigma}$ is not equal to zero. Hence w_1 is a homogeneous linear function of X, Y, Z and W, when the lines joining corresponding points on Σ and Σ_1 are concurrent. Conversely, suppose that w_1 is a homogeneous linear function of the form of the left-hand member of (14). Apply to Σ the polar transformation with respect to the quadric (5) and let S be the transform of Σ. Take an associate surface of S, say S_1. When now the transformation with respect to the quadric (5) is applied to S_1, we get a surface Σ_1 in the relation of a transformation Ω to Σ, the function w_1 differing by a constant factor at most from the left-hand member of (14). Combining this result with the observations made in the preceding paragraph, we have the theorem:

When the function w_1 determining a transformation Ω of a surface Σ is equal to W plus a homogeneous linear function of the direction-cosines of the normal to Σ, ths lines joining corresponding points on Σ and its transform Σ_1 are concurrent; and conversely.

If in (5) we put $c = 0$, we have in place of (14) the equation $Z = f/\sqrt{\sigma}$. Consequently w_1 differs from Z by a constant factor at most, which is unessential. Since the z coördinate of the plane at infinity with respect to the quadric (5) is infinite, the lines joining corresponding points on Σ and Σ_1 are parallel. By reasoning analogous to the preceding we arrive at the theorem:

When the function w_1 determining a transformation Ω of a surface Σ is a homogeneous linear function of the direction-cosines of the normal to Σ, the lines joining the points on Σ and its transform Σ_1 are parallel; and conversely.

If θ_1 is a linear function of x, y, z, it is possible to choose the quadric (5) so that by means of (7) and (14) $\theta_1/\sqrt{\sigma}$ is expressible as a homogeneous linear function of X, Y, Z, W, involving W at least. Hence:

When the function θ_1 determining a transformation K of a surface S is a linear function of the cartesian coördinates of S, the corresponding tangent planes to S and its transform meet in line of a fixed plane.

[1] Transformations of conjugate systems with equal point invariants, *Trans. Amer. Math. Soc.*, **15**, 397–430 (1914). This will be referred to as memoir M_1.

[2] Conjugate systems with equal tangential invariants and the transformation of Moutard, *Palermo, Rend. Circ. Mat.*, **39**, (1915). This will be referred to as memoir M_2.

[3] Guichard, *Ann. sci. Ec. norm., Paris*, Ser. 3, **14**, 492 (1897).

ON THE POLE EFFECT IN THE IRON ARC

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In a communication to the Academy which appeared in the PROCEEDINGS for March 1915, we reported some results of our investigations on the pole effect in the iron arc under normal conditions. We have found between two and three hundred lines whose wave-lengths at the negative pole are distinctly longer than when the light is taken from a cross section at the center of the arc.

Aside from the theoretical interest in such changes in wave-length, reference may be made to the following points:

1. A number of these lines are included among the international standards of the second order. Their wave-lengths depend upon interferometer measurements made by three independent observers, the means of which have been adopted as standards by the International Union for Coöperation in Solar Research.

2. There is a region of the iron spectrum extending from λ 5500 to λ 6000 in which no other class of lines is available for standards.

3. In various laboratories there are in progress redeterminations, based upon the iron standards, of the wave-lengths in international units of the lines of many elements. In these redeterminations the instrument most frequently used is the concave grating in the usual Rowland mounting. In ordinary practice, the slit of the spectrograph is parallel to the axis of the arc and includes the major portion of its length. We have found that the pole effect appears at a considerable distance from the negative pole and that for high precision